The evaluation of weight multiplicities of G_2

1978 J. Phys. A: Math. Gen. 11 1491

(http://iopscience.iop.org/0305-4470/11/8/010)

J. Phys. A: Math. Gen. Vol. 11, No. 8, 1978. Printed in Great Britain. © 1978

The evaluation of weight multiplicities of G_2

R C King and A H A Qubanchi
Mathematics Department, The University, Southampton, SO9 5NH, UK

Received 11 January 1978

Abstract. The branching rule given by Fronsdal for the restriction from G_2 to SU(3) is used to obtain the multiplicities of the dominant weights of representations of G_2 by means of a recurrence relation having a simple geometric interpretation. The branching multiplicities and weight multiplicities are tabulated for the representations $\{\mu_1\ \mu_2\}$ of G_2 with $0 \leqslant \mu_1 + \mu_2 \leqslant 9$.

1. Introduction

The branching rule for the restriction from G_2 to SU(3) was first derived by Fronsdal (1962) and subsequently re-derived using different methods by Sharp and Lam (1969), Sviridov *et al* (1973) and Perroud (1976). The result is remarkably simple but has not previously been exploited in order to calculate the weight multiplicities of G_2. Instead a variety of other methods of calculating these multiplicities have been developed by Antoine and Speiser (1964), Radhakrishnan and Santhanam (1967), Gruber and Weber (1968), McConnell (1968), Springer (1968), and Gruber (1970). These methods are all rather complicated and the only tabulation of results is that of Springer. In this paper explicit expressions introduced elsewhere (King and Qubanchi 1978) for the characters of irreducible representations of G_2 and SU(3) are used to outline Fronsdal's derivation of the branching rule appropriate to the restriction from G_2 to SU(3). This is done with a view to simplifying the calculation of weight multiplicities of G_2 by exploiting the well known results for the weight multiplicities of SU(3) due originally to Wigner (1937). A very simple recurrence relation is obtained and a number of results tabulated.

2. Characters of SU(3)

The irreducible representation $\{\nu_1\ \nu_2\}$ of SU(3) has character:

$$\text{SU(3):} \qquad \chi^{\{\nu_1\ \nu_2\}} = \frac{\sum_{\pi \in S_3} (-1)^{\pi} x_{\pi_1}^{2+\nu_1} x_{\pi_2}^{1+\nu_2} x_{\pi_3}^{0}}{\sum_{\pi \in S_3} (-1)^{\pi} x_{\pi_1}^{2} x_{\pi_2}^{1} x_{\pi_3}^{0}}, \tag{2.1}$$

with the class parameters constrained by the condition

$$x_1 x_2 x_3 = 1. \tag{2.2}$$

The permutation π of S_3 denotes the map from (1 2 3) to $(\pi_1\ \pi_2\ \pi_3)$, and $(-1)^{\pi}$

denotes the parity of π. This character may be rewritten in the form:

$$\text{SU(3)}: \qquad \chi^{\{\nu_1\,\nu_2\}} = \sum_{n_1,n_2,n_3} K^{\{\nu_1\,\nu_2\}}_{(n_1\,n_2\,n_3)} x_1^{n_1} x_2^{n_2} x_3^{n_3} \tag{2.3}$$

where the coefficient $K^{\{\nu_1\,\nu_2\}}_{(n_1\,n_2\,n_3)}$ is the multiplicity of the weight $(n_1\,n_2\,n_3)$ in the representation $\{\nu_1\,\nu_2\}$. The corresponding weight diagram is the point set in which each point specified by the triangular coordinates $(n_1\,n_2\,n_3)$ is assigned the multiplicity

$$K^{\{\nu_1\,\nu_2\}}_{(n_1\,n_2\,n_3)} = 1 + \min(\nu_1-\nu_2,\, \nu_2,\, \nu_1-n_1,\, \nu_1-n_2,\, \nu_1-n_3,\, n_1,\, n_2,\, n_3) \tag{2.4}$$

where

$$\min(\kappa_1, \kappa_2, \ldots) = \begin{cases} \text{minimum of } \kappa_1, \kappa_2, \ldots & \text{if } \kappa_i \geqslant 0 \quad \text{for all } i, \\ -1 & \text{if } \kappa_i < 0 \text{ for any } i. \end{cases}$$

The use of triangular coordinates in a plane is dictated by the constraint (2.2) and implies the coincidence of points labelled by $(n_1\,n_2\,n_3)$ and $(n_1+n_0,\, n_1+n_0,\, n_3+n_0)$ for any n_0. The weight diagrams have the well known hexagonal structure pointed out by Wigner (1937).

Both the numerator and the denominator of the expression (2.1) for the character of the SU(3) representation $\{\nu_1\,\nu_2\}$ may also define point sets. The quotient of these point sets is thus the weight diagram of $\{\nu_1\,\nu_2\}$.

3. The branching rule for $G_2 \downarrow$ SU(3)

The formula analogous to (2.1) for the character of the irreducible representation $\{\mu_1\,\mu_2\}$ of G_2 is:

$$G_2: \qquad \chi^{\{\mu_1\,\mu_2\}} = \frac{\sum_{\pi\in S_3} (-1)^\pi (x_{\pi_1}^{2+\mu_1} x_{\pi_2}^{0} x_{\pi_3}^{-1-\mu_2} + x_{\pi_1}^{-2-\mu_1} x_{\pi_2}^{0} x_{\pi_3}^{1+\mu_2})}{\sum_{\pi\in S_3} (-1)^\pi (x_{\pi_1}^{2} x_{\pi_2}^{0} x_{\pi_3}^{-1} + x_{\pi_1}^{-2} x_{\pi_2}^{0} x_{\pi_3}^{1})}. \tag{3.1}$$

Once again the class parameters are constrained by the condition (2.2). Furthermore this parametrisation is such that the branching rule appropriate to the restriction of group elements from G_2 to SU(3) may be obtained merely by writing the character (3.1) as a linear combination of the characters (2.1):

$$G_2 \downarrow \text{SU(3)}: \quad \chi^{\{\mu_1\,\mu_2\}} = \sum_{\nu_1 \geqslant \nu_2 \geqslant 0} C^{\{\mu_1\,\mu_2\}}_{\{\nu_1\,\nu_2\}} \chi^{\{\nu_1\,\nu_2\}}. \tag{3.2}$$

The branching multiplicity diagram may conveniently be constructed as a point set in which each point specified by the oblique coordinates $(\nu_1\,\nu_2)$ is assigned the branching multiplicity $C^{\{\mu_1\,\mu_2\}}_{\{\nu_1\,\nu_2\}}$.

Making use of (2.2) in (3.1) it is clear that

$$G_2: \qquad \chi^{\{\mu_1\,\mu_2\}} = \frac{\sum_{\pi\in S_3} (-1)^\pi (x_{\pi_1}^{3+\mu_1+\mu_2} x_{\pi_2}^{1+\mu_2} x_{\pi_3}^{0} - x_{\pi_1}^{3+\mu_1+\mu_2} x_{\pi_2}^{2+\mu_1} x_{\pi_3}^{0})}{\sum_{\pi\in S_3} (-1)^\pi (x_{\pi_1}^{3} x_{\pi_2}^{1} x_{\pi_3}^{0} - x_{\pi_1}^{3} x_{\pi_2}^{2} x_{\pi_3}^{0})}. \tag{3.3}$$

Comparison with (2.1) yields the formula:

$$\chi^{\{\mu_1\,\mu_2\}} = \frac{\chi^{\{\mu_1+\mu_2+1,\mu_2\}} - \chi^{\{\mu_1+\mu_2+1,\mu_1+1\}}}{\chi^{\{1,0\}} - \chi^{\{1,1\}}}. \tag{3.4}$$

In order to evaluate this quotient it is merely necessary to notice that the denominator is

$$x_1+x_2+x_3-x_1^{-1}-x_2^{-1}-x_3^{-1},$$

where use has been made of (2.2), whilst the numerator is:

$$\chi^{\{\mu_1+\mu_2+1,\mu_2\}}-\chi^{\{\mu_1+\mu_2+1,\mu_1+1\}}+\chi^{\{\mu_1,\mu_1+1\}}-\chi^{\{\mu_2-1,\mu_2\}}+\chi^{\{\mu_2-1,-1\}}-\chi^{\{\mu_1,-1\}}$$

by virtue of the validity of the modification rules:

$$\mathrm{SU}(3): \qquad \chi^{\{\nu_1\,\nu_2\}}=-\chi^{\{\nu_2-1,\nu_1+1\}}=-\chi^{\{\nu_1-\nu_2-1,-\nu_2-2\}} \tag{3.5}$$

which follow immediately from the character formula (2.1) and which imply that the last four terms all vanish. It is then straightforward to see that the point sets defined by the numerator and denominator of (3.4) are isomorphic to those defined by the numerator and denominator of (2.1). It follows that the quotient point set for (3.4) is isomorphic to the weight diagram of the representation $\{\mu_1\,\mu_2\}$ of SU(3) in the space of irreducible representations of SU(3) with points specified by oblique coordinates $(\nu_1\,\nu_2)$. More precisely

$$C^{\{\mu_1\,\mu_2\}}_{\{\nu_1\,\nu_2\}}=K^{\{\mu_1\,\mu_2\}}_{(\nu_1-\nu_2,\nu_2,\mu_1+\mu_2-\nu_1)}, \tag{3.6}$$

so that the branching multiplicities for G_2 restricted to SU(3) are seen to be nothing other than weight multiplicities of SU(3). This result was pointed out by Fronsdal (1962), who remarked on the 'strange' nature of this relationship between G_2 and SU(3).

An illustrative example is given in figure 1 for the case $\{\mu_1\,\mu_2\}=\{5\ 2\}$ and the results appropriate to the representations $\{\mu_1\,\mu_2\}$ with $0\leqslant\mu_1+\mu_2\leqslant 9$ are given in table 1.

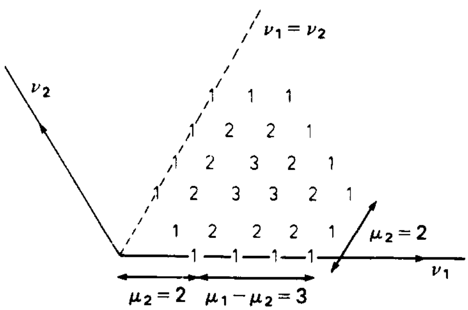

Figure 1. The branching multiplicity diagram for $G_2\downarrow$SU(3) for the representation $\{\mu_1\,\mu_2\}=\{5\ 2\}$ of G_2 into irreducible representations $\{\nu_1\,\nu_2\}$ of SU(3). The numbers in the array are the multiplicities $C^{\{5\,2\}}_{\{\nu_1\,\nu_2\}}$.

4. The weight multiplicities of G_2

The multiplicity $M^{\{\mu_1\,\mu_2\}}_{(n_1\,n_2\,n_3)}$ of the weight $(n_1\,n_2\,n_3)$ in the representation $\{\mu_1\,\mu_2\}$ of G_2 is defined by the expansion:

$$G_2: \qquad \chi^{\{\mu_1\,\mu_2\}}=\sum_{(n_1\,n_2\,n_3)} M^{\{\mu_1\,\mu_2\}}_{(n_1\,n_2\,n_3)}x_1^{n_1}x_2^{n_2}x_3^{n_3}. \tag{4.1}$$

Table 1. Branching multiplicities $C^{\{\mu_1 \mu_2\}}_{\{\nu_1 \nu_2\}}$ of the representations $\{\mu_1 \mu_2\}$ of G_2 into representations $\{\nu_1 \nu_2\}$ of SU(3).

$\{\nu_1 \nu_2\}$		11		22	32	33		43	44	53	54	55		64	65	66	74	75	76	77		85	86	87	88	95	96	97	98	99
$\{\mu_1 \mu_2\}$	00	10	21	20	31	30	42	41	40	52	51	50	63	62	61	60	73	72	71	70	84	83	82	81	80	94	93	92	91	90
00	1																													
10	1	1																												
11		1	1																											
20	1	1	1	1																										
21		1	2	1	1																									
30	1	1	1	1	1	1																								
22			1	1	1		1																							
31		1	2	1	2	1	1	1																						
40	1	1	1	1	1	1	1	1	1																					
32			1	1	2	1	2	1		1																				
41		1	2	1	2	1	2	2	1	1	1																			
50	1	1	1	1	1	1	1	1	1	1	1	1																		
33					1	1	1	1		1			1																	
42			1	1	2	1	3	2	1	2	1		1	1																
51		1	2	1	2	1	2	2	1	2	2	1	1	1	1															
60	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1														
43					1	1	2	2	1	2	1		2	1			1													
52			1	1	2	1	3	2	1	3	2	1	2	2	1		1	1												
61		1	2	1	2	1	2	2	1	2	2	1	2	2	2	1	1	1	1											
70	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1										
44							1	1	1	1	1		1	1			1				1									
53					1	1	2	2	1	3	2	1	3	2	1		2	1			1	1								
62			1	1	2	1	3	2	1	3	2	1	3	3	2	1	2	2	1		1	1	1							
71		1	2	1	2	1	2	2	1	2	2	1	2	2	2	1	2	2	2	1	1	1	1	1						
80	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1					
54							1	1	1	2	2	1	2	2	1		2	1			2	1				1				
63					1	1	2	2	1	3	2	1	4	3	2	1	3	2	1		2	2	1			1	1			
72			1	1	2	1	3	2	1	3	2	1	3	3	2	1	3	3	2	1	2	2	2	1		1	1	1		
81		1	2	1	2	1	2	2	1	2	2	1	2	2	2	1	2	2	2	1	2	2	2	2	1	1	1	1	1	
90	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Combining (3.2), (3.6) and (2.3) yields the formula:

$$M^{\{\mu_1\ \mu_2\}}_{(n_1\ n_2\ n_3)} = \sum_{\nu_1 \geq \nu_2 \geq 0} K^{\{\mu_1\ \mu_2\}}_{(\nu_1-\nu_2, \nu_2, \mu_1+\mu_2-\nu_1)} K^{\{\nu_1\ \nu_2\}}_{(n_1\ n_2\ n_3)}. \tag{4.2}$$

The Weyl symmetry manifested by the character formula (3.1) leads to a symmetry of the corresponding weight diagram whereby each weight vector $(n_1\ n_2\ n_3)$ is related by the Weyl group to a dominant weight for which $n_1 - n_2 = m_1 \geq n_2 - n_3 = m_2 \geq 0$. The multiplicity of such a dominant weight is more conveniently denoted by:

$$M^{\{\mu_1\ \mu_2\}}_{(m_1\ m_2)} = \sum_{\nu_1 \geq \nu_2 \geq 0} K^{\{\mu_1\ \mu_2\}}_{(\nu_1-\nu_2, \nu_2, \mu_1+\mu_2-\nu_1)} K^{\{\nu_1\ \nu_2\}}_{(m_1+m_0, m_0, m_0-m_2)} \tag{4.3}$$

where $m_1 \geq m_2 \geq 0$ and $3m_0 + m_1 - m_2 = \nu_1 + \nu_2$, with $m_0 \geq m_2$.

The first factor, as has been explained in § 3, defines the hexagonal point set associated with the representation $\{\mu_1\ \mu_2\}$ of SU(3) in which the points are labelled by oblique coordinates $(\nu_1\ \nu_2)$. The contribution to e dominant weight $(m_1\ m_2)$ of the representation $\{\nu_1\ \nu_2\}$ of SU(3) is then found from the second factor.

This contribution may be evaluated by noting that for the dominant weights $m_1 + m_0 \geq m_0 \geq m_0 - m_2$ so that by virtue of (2.4)

$$K^{\{\nu_1\ \nu_2\}}_{(m_1+m_2, m_0 m_0-m_2)} = 1 + \min(\nu_1 - \nu_2,\ \nu_2,\ \nu_1 - m_1 - m_0,\ m_0 - m_2), \tag{4.4}$$

where the four arguments of min(. . .) are the distances of the point $(\nu_1\ \nu_2)$ from the four lines

$$\nu_1 - \nu_2 = 0, \qquad \nu_2 = 0, \qquad 2\nu_1 - \nu_2 = 2m_1 + m_2 \qquad \text{and} \qquad \nu_1 + \nu_2 = m_1 + 2m_2.$$

The corresponding point set is bounded by these four lines with multiplicity 1 at points on this boundary and increasing indefinitely in steps of 1 along lines parallel to $\nu_1 = 2\nu_2$. This is illustrated for the case $(m_1\ m_2) = (2\ 1)$ in figure 2.

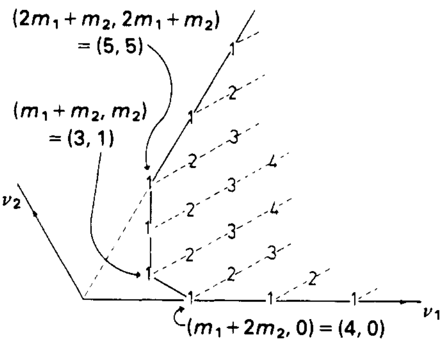

Figure 2. The point set defined by $K^{\{\nu_1\ \nu_2\}}_{(m_1+m_0, m_0, m_0-m_2)}$ for the case $(m_1\ m_2) = (2\ 1)$ with $3m_0 + 1 = \nu_1 + \nu_2$.

The multiplicity $M^{\{\mu_1\ \mu_2\}}_{(m_1\ m_2)}$ is then found by superposing the point sets appropriate to the two factors of (4.3) and summing the pairwise products of the contributions from each point. This yields, as shown in figure 3 for the case of the representation $\{\mu_1\ \mu_2\} = \{5\ 2\}$ and the dominant weight $(m_1\ m_2) = (2\ 1)$, the multiplicity:

$$M^{\{5\ 2\}}_{(2\ 1)} = 1.1 + 1.2 + 2.1 + 3.2 + 1.3 + 2.1 + 2.2 + 1.1 = 21$$

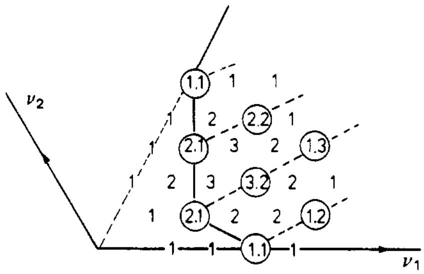

Figure 3. The evaluation of $M^{\{\mu_1\,\mu_2\}}_{(m_1\,m_2)}$ in the case $\{\mu_1\,\mu_2\}=\{5\,2\}$ and $(m_1\,m_2)=(2\,1)$ by the superposition of the weight diagram of $\{5\,2\}$ on the point set $K^{\{\nu_1\,\nu_2\}}_{(2+m_0,m_0,m_0-1)}$. The result is the sum of the non-vanishing products which are encircled, so that $M^{\{5\,2\}}_{(2\,1)}=21$.

This same procedure may be followed for each point in turn or it may be noted from the structure of the point sets that:

$$M^{\{\mu_1\,\mu_2\}}_{(m_1\,m_2)}=M^{\{\mu_1\,\mu_2\}}_{(m_1+1,m_2+1)}+R^{\{\mu_1\,\mu_2\}}_{(m_1\,m_2)} \tag{4.5}$$

where

$$R^{\{\mu_1\,\mu_2\}}_{(m_1\,m_2)}=\sum_{\nu_1\geqslant0,\nu_2\geqslant0}K^{\{\mu_1\,\mu_2\}}_{(\nu_1-\nu_2,\nu_2,\mu_1+\mu_2-\nu_1)} \tag{4.6}$$

with the summation carried out over all points $(\nu_1\,\nu_2)$ for which

$$(\nu_1\,\nu_2)=\begin{cases}(m_1+m_2+\alpha+2\gamma,\ m_2+2\alpha+\gamma) & \text{for } 0\leqslant\alpha\leqslant m_1\\ (m_1+m_2+\beta+2\gamma,\ m_2-\beta+\gamma) & \text{for } 0\leqslant\beta\leqslant m_2\end{cases}$$

with $\gamma=0,1,2,\ldots$.

Diagramatically this recurrence relation has a simple interpretation whereby the contribution to $R^{\{\mu_1\,\mu_2\}}_{(m_1\,m_2)}$ is just the point set defined by the intersection of the weight diagram of $\{\mu_1\,\mu_2\}$ with the lines parallel to $\nu_1=2\nu_2$ through the points on that part of the boundary of the point set defined by the second factor of (4.3) specified by $2\nu_1-\nu_2=2m_1+m_2$ and $\nu_1+\nu_2=m_1+2m_2$. No multiplication is required. This is exemplified in figure 4 in the case $\{\mu_1\,\mu_2\}=\{5\,2\}$ and $(m_1\,m_2)=(2\,1)$ and in figure 5 in the case $\{\mu_1\,\mu_2\}=\{5\,2\}$ and $(m_1\,m_2)=(1\,0)$.

Since $M^{\{5\,2\}}_{(3\,2)}=8$ it follows from these diagrams that $M^{\{5\,2\}}_{(2\,1)}=M^{\{5\,2\}}_{(3\,2)}+13=21$ in agreement with the previous calculation, and $M^{\{5\,2\}}_{(1\,0)}=M^{\{5\,2\}}_{(2\,1)}+11=32$.

Clearly the application of the recurrence relation (4.5) commencing with points $(m_1\,m_2)$ at the outer edge of the weight diagram enables the multiplicities of all

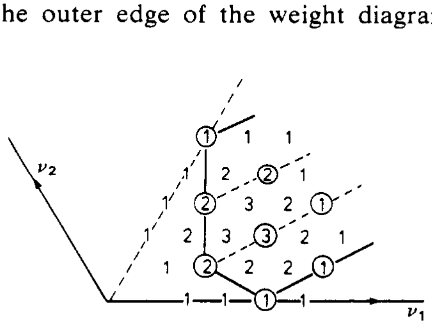

Figure 4. The evaluation of $R^{\{\mu_1\,\mu_2\}}_{(m_1\,m_2)}$ in the case $\{\mu_1\,\mu_2\}=\{5\,2\}$ and $(m_1\,m_2)=(2\,1)$ through the intersection of the weight diagram of $\{5\,2\}$ and the relevant points of the set $K^{\{\nu_1\,\nu_2\}}_{(2+m_0,m_0,m_0-1)}$. The result is the sum of the encircled numbers, so that $R^{\{5\,2\}}_{(2\,1)}=13$.

Table 2. The multiplicities $M^{\{\mu_1\,\mu_2\}}_{(m_1\,m_2)}$ of the dominant weights $(m_1\,m_2)$ of the representations $\{\mu_1\,\mu_2\}$ of G_2.

$\{\mu_1\,\mu_2\}$ \ $(m_1\,m_2)$	00	10	11	20	21	30	22	31	40	32	41	50	33	42	51	60	43	52	61	70	44	53	62	71	80	54	63	72	81	90
00	1																													
10	1	1																												
11	2	1	1																											
20	3	2	1	1																										
21	4	4	2	2	1																									
30	5	4	3	2	1	1																								
22	5	3	3	2	1	1	1																							
31	9	8	6	5	3	2	1	1																						
40	8	7	5	5	3	2	1	1	1																					
32	10	10	7	7	5	3	2	2	1	1																				
41	16	14	12	10	7	6	4	3	2	1	1																			
50	12	11	9	8	6	5	3	3	2	1	1	1																		
33	9	7	7	5	4	4	3	2	1	1	1	0	1																	
42	21	19	16	15	11	9	7	6	4	3	2	1	1	1																
51	24	23	19	18	14	11	8	8	6	4	3	2	1	1	1															
60	18	16	14	13	10	9	7	6	5	3	3	2	1	1	1	1														
43	20	20	16	16	13	10	8	8	6	5	3	2	2	2	1	0	1													
52	35	32	29	26	21	19	15	13	10	8	7	4	4	3	2	1	1	1												
61	36	34	30	28	23	20	16	15	12	9	8	6	4	4	3	2	1	1	1											
70	24	23	20	19	16	14	11	11	9	7	6	5	3	3	3	2	1	1	1	1										
44	16	13	13	11	9	9	8	6	5	4	4	2	3	2	1	1	1	1	0	0	1									
53	39	37	33	31	26	23	19	18	14	12	10	7	7	6	4	2	3	2	1	0	1	1								
62	52	50	44	43	36	31	26	25	21	17	14	11	9	9	7	4	4	3	2	1	1	1	1							
71	51	48	44	41	35	32	27	25	21	17	16	12	10	9	8	6	4	4	3	2	1	1	1	1						
80	33	31	28	27	23	21	18	17	15	12	11	9	7	7	6	5	3	3	3	2	1	1	1	1	1					
54	35	35	30	30	26	22	19	19	16	14	11	9	8	8	6	3	5	3	2	1	2	2	1	0	0	1				
63	64	60	56	52	45	42	36	33	28	24	22	16	16	14	11	8	8	7	4	2	4	3	2	1	0	1	1			
72	75	72	66	63	55	50	43	41	35	30	27	22	19	18	15	11	10	9	7	4	4	4	3	2	1	1	1	1		
81	68	66	60	58	51	46	40	39	34	29	26	22	18	18	16	12	10	9	8	6	4	4	4	3	2	1	1	1	1	
90	43	41	38	36	32	30	26	25	22	19	18	15	13	12	11	9	7	7	6	5	3	3	3	3	2	1	1	1	1	1

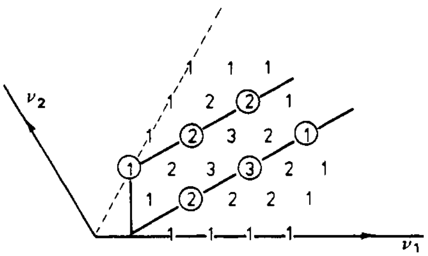

Figure 5. The evaluation of $R^{\{\mu_1\,\mu_2\}}_{(m_1\,m_2)}$ in the case $\{\mu_1\,\mu_2\}=\{5\ 2\}$ and $(m_1\,m_2)=(1\ 0)$ through the intersection of the weight diagram of {5 2} and the relevant points of the set $K^{\{\nu_1\,\nu_2\}}_{(1+m_0,m_0,m_0)}$. The result is the sum of the encircled numbers, so that $R^{\{5\ 2\}}_{(1\ 0)}=11$.

dominant weights to be evaluated. The results appropriate to the representation $\{\mu_1\,\mu_2\}=\{5\ 2\}$ are displayed in figure 6.

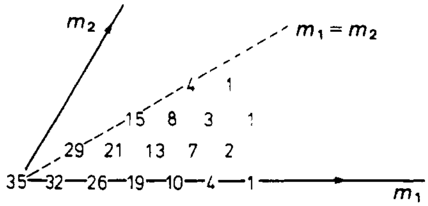

Figure 6. The multiplicities $M^{\{5\ 2\}}_{(m_1\,m_2)}$ of the dominant weights $(m_1\,m_2)$ of the representation {5 2} of G_2.

Finally the complete weight diagram may be constructed through the Weyl symmetry operations. For the simpler representation $\{\mu_1\,\mu_2\}=\{3\ 1\}$ this is shown in figure 7.

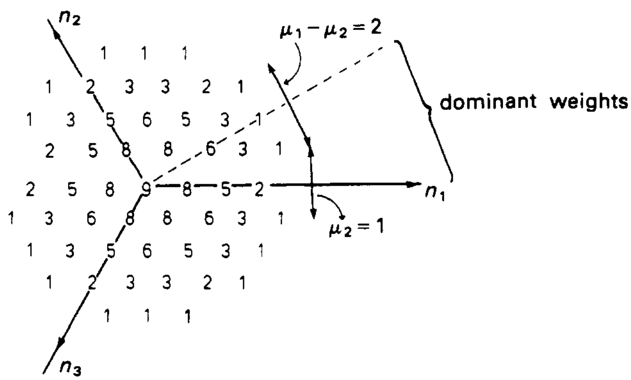

Figure 7. The weight multiplicity diagram for the representation $\{\mu_1\,\mu_2\}=\{3\ 1\}$ of G_2. The numbers displayed are the multiplicities $M^{\{3\ 1\}}_{(n_1\,n_2\,n_3)}$.

The twelve-fold symmetry is evident whereby the complete diagram may be generated from the dominant weight multiplicities. The results for the multiplicities of the dominant weights of the representations $\{\mu_1\,\mu_2\}$ of G_2 with $0\leqslant\mu_1+\mu_2\leqslant 9$ are given in table 2, extending the tabulation of Springer (1968).

5. Conclusion

It should be stressed that the important aspect of the development presented here is the use of point set diagrams to describe both weight multiplicities and branching multiplicities. This together with the especially simple rules for evaluating the weight multiplicities of SU(3) lead very easily to the derivation of the crucial recurrence relation (4.5) for dominant weight multiplicities. The simple form of this relation obviates the need to cope with the more complex calculations referred to in the introduction.

References

Antoine J P and Speiser D 1964 *J. Math. Phys.* **5** 1560–72
Fronsdal C 1962 *Elementary Particle Physics and Field Theory: Brandeis Summer Institute 1962* (New York: Benjamin) pp 427–532
Gruber B 1970 *J. Math. Phys.* **11** 3077–90
Gruber B and Weber H J 1968 *Proc. R. Irish Acad.* A **66** 31–40
King R C and Qubanchi A H A 1978 *J. Phys. A: Math. Gen.* **11** 1–7
McConnell J 1968 *Proc. R. Irish Acad.* A **66** 72–92
Perroud M 1976 *J. Math. Phys.* **17** 1998–2006
Radhakrishnan D and Santhanam T S 1967 *J. Math. Phys.* **8** 2206–9
Sharp R T and Lam C S 1969 *J. Math. Phys.* **10** 2033–8
Springer T A 1968 *Invent. Math.* **5** 85–105
Sviridov D T, Smirnov Yu F and Tolstoi V N 1973 *Sov. Phys.–Dokl.* **17** 976–8
Wigner E P 1937 *Phys. Rev.* **51** 106–19